White Paper

Interest Rate Swap Valuation

Tom P. Davis, PhD

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Contents

Introduction

An interest rate swap is a legal contract entered into by two parties to exchange cash flows on an agreed upon set of future dates. The interest rate swaps market constitutes the largest and most liquid part of the global derivatives market. At the end of June 2014, the total notional amount of outstanding contracts was $563 trillion, representing 81% of the over-the-counter global derivatives market, and the gross market value of interest rate derivatives totaled $13 trillion.[1] The focus of this paper is on plain vanilla swaps, which constitute the vast majority of the OTC swap market.

Each stream of cash flows is referred to as a “leg.” A plain vanilla interest rate swap has two legs – a fixed leg and a floating leg. The fixed leg cash flows are set when the contract is initiated, whereas the floating leg cash flows are determined on “rate fixing dates,” which occur close to the beginning of the payment period and are specified as part of the contract terms and conditions.

The current market value of an interest rate swap is determined by the prevailing interest rate environment on the valuation date, represented by the set of current interest rate curves. There are two important curves for valuing interest rate swaps – the overnight curve and the floating rate index curve relevant to the jurisdiction, which for plain vanilla swaps is the Interbank Offered Rate (IBOR).

In this paper, IBOR will refer to a generic interbank offered rate. When a specific IBOR is referenced, such as the USD London Interbank Offered Rate (LIBOR), the more specific name will be used (i.e., USD LIBOR). For instance, this distinction becomes important to differentiate between Euribor®, which is the interbank offered rate for the euro set by banks in the Eurozone, and EUR LIBOR, which is the interbank offered rate for the euro set by banks in London.

1 Bank for International Settlements, “OTC Derivatives Statistics at End-June 2014”, 2014, https://www.bis.org/publ/otc_hy1411.pdf.

The correct curves depend on the jurisdiction in which the swap is being valued, as shown in Table 1:

Table 1: Curves by Jurisdiction[2]

Jurisdiction	Relevant Overnight Curve	Overnight Day Count[3]
USD	U.S. Federal Reserve (federal funds rate)	ACT/360
GBP	Sterling Overnight Index Average (SONIA)	ACT/365
JPY	Tokyo Overnight Average Rate (TONAR, also called MUTAN)	ACT/365
EUR	Euro Overnight Index Average (Eonia®)	ACT/360
CHF	Swiss Average Rate Overnight (SARON®)	ACT/360

A crucial difference between the overnight rates and IBOR indices is that overnight rates are averages of uncollateralized overnight lending rates that actually transact in the market, not averages of hypothetical funding rates that do not transact. The market tends to show that the overnight rates are more representative of the lowest credit risk; therefore, the overnight rate is the closest proxy to the risk-free rate. Furthermore, since the overnight rate is representative of an actual market transaction, it is less prone to manipulation by the participants that determine the rate. LIBOR rates are set by a polling procedure at 11:00 a.m. GMT each day. USD LIBOR rates are determined by asking 18 of the member banks what rate they could receive to borrow uncollateralized funds in the interbank market. The calculation agent collects the quotes and removes the top four and bottom four. The average of the remaining 10 quotes defines LIBOR and is then published by market data providers.

The overnight rate curve is constructed using a bootstrapping and interpolation process based on the liquidly traded market instruments that reference the overnight rate, usually Overnight Index Swaps (OIS) instruments.

The IBOR curve is constructed using a bootstrapping and interpolation process based on the prices of liquidly traded contracts that reference the IBOR rate. In order of maturity, these instruments are cash deposits up to around one year, followed by Eurodollar futures with maturities up to two years, and interest rate swaps with maturities from two years to 30 years. FactSet obtains these market data quotes daily from Tullett Prebon.

2 OpenGamma, 'Interest Rate Instruments and Market Conventions Guide', 2013, http://www.opengamma.com/sites/default/files/interest-rate-instruments-and-market-conventions.pdf.
3 See 2006 ISDA Definitions. (New York, NY: International Swaps and Derivatives Association, Inc., 2006) for comprehensive descriptions of day count conventions.

History of the Projecting and Discounting Curves

Prior to 2007, both the projecting index curve and the discounting index curve could be calculated using a unique curve. An index rate was related to the discount factor curve by an arbitrage relationship. The index rate was determined to be the unique simple rate of interest relating the zero coupon bonds between the accrual start date and the accrual end date (equivalently the rate start date and the rate maturity). To avoid arbitrage, the rate can be inferred from the discount curve by setting up a portfolio that is initially costless.

Equation 1 shows how to calculate the expected index rate determined by the current interest rate environment:

Equation 1:

$$\frac{D(t,T_i) - D(t,T_i+\tau)}{D(t,T_i+\tau)} = \tau L_i^{\tau}(t) = Index\ Rate$$

Where:

- $D(t,T_i)$ = The discount factor, representing the price of a zero coupon bond at the time it is bought (t) and the time it matures (T_i)
- τ = The tenor of the index rate, usually three months or six months depending on the jurisdiction

This equation served as the definition of the index rate and is usually written in a different form, as shown in Equation 2:

Equation 2:

$$Index\ Rate^4 = \tau L_i^{\tau}(t) = \frac{1}{\tau}\left(\frac{D(t,T_i)}{D(t,T_i+\tau)} - 1\right)$$

Where:

- $D(t,T_i)$ = The discount factor, representing the price of a zero coupon bond at the time it is bought $(t)$and the time it matures (T_i)
- τ = The tenor of the index rate, usually three months or six months depending on the jurisdiction

In 2008, the world changed. The ability to determine the index rate from the discount curve in Equation 1 was no longer valid. In other words, the same curve could no longer be used to project future cash flows and to discount future cash flows as evidenced by the persistent non-zero basis spreads.

In this post-crisis world, Equation 2 was modified by adding a spread, as shown in Equation 3:

Equation 3:

$$Index\ Rate = L_i^{\tau}(t) = \frac{1}{\tau}\left(\frac{D(t,T_i)}{D(t,T_i+\tau)} - 1\right) + \kappa(t)$$

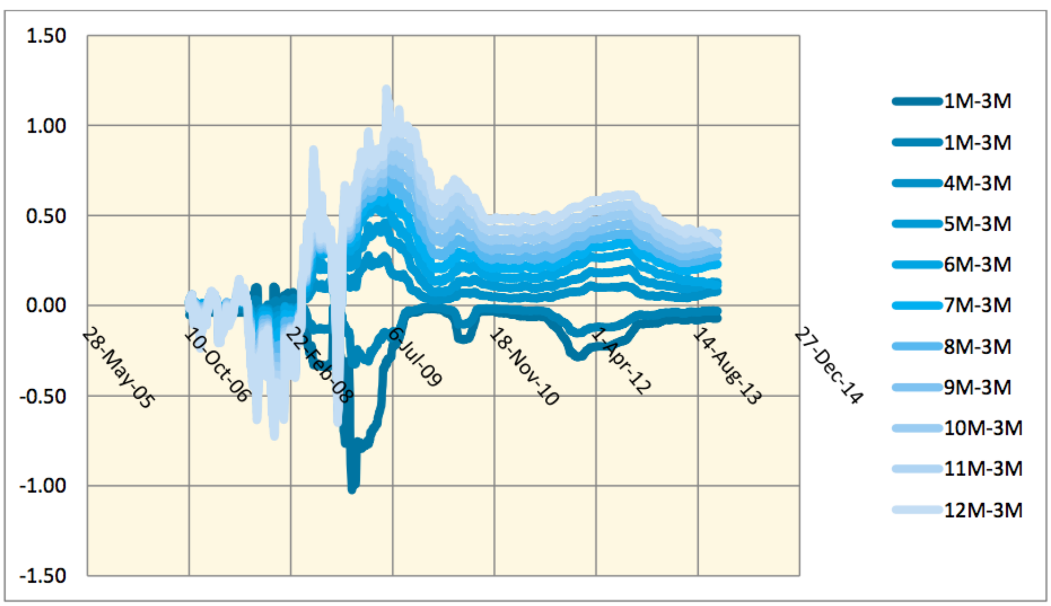
Where:

- $\kappa^{\tau}(t)$ = A tenor (τ) dependent piecewise constant spread
- $DD(t,T_i)$= The discount factor, representing the price of a zero coupon bond at the time it is bought $(t)$and the time it matures (T_i)
- τ = The tenor of the index rate, usually three months or six months depending on the jurisdiction

4 The tau superscript on $L_i^{\tau}(t)$ is usually known from context and can be omitted.

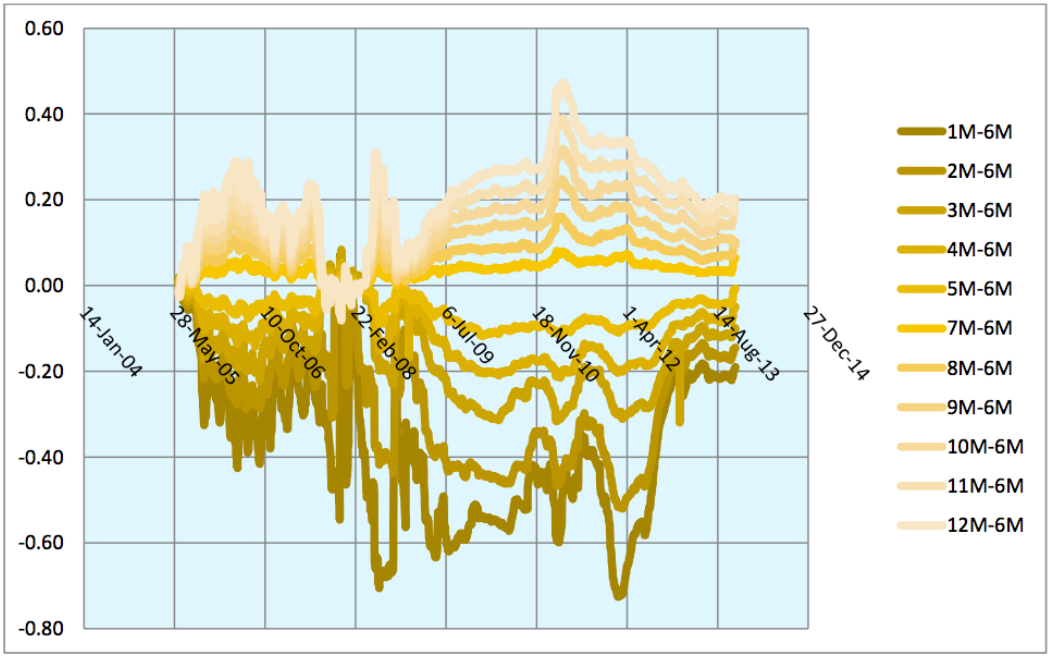
A spread can arise in two ways, both having the same financial fundamental origin. First, as a liquidity premium reflected in the IBOR index rates and second as a credit spread. All things being equal, an investor would rather receive payments more frequently;[5] the reason is credit worthiness of the institutions quoting IBOR. Prior to 2007, the IBOR index rates did not account for the possibility of major banks defaulting; therefore, the rates did not reflect a credit spread. The events in 2007-2008 showed that it was unreasonable to assuming top-tier banks could not default. The spreads all demonstrate the correct behavior as inferred by the credit argument that lower tenors will trade at spreads lower than higher tenors due to the decreased time over which the lending took place. This is carried out in both the USD and EUR markets as shown in Figure 1 and Figure 2.

Figure 1: USD LIBOR Spreads versus 6M



5 LIBOR rates set the magnitude of the swap floating rate cash flows; therefore, the intuition is that the market demanding a premium will serve to increase the rate. This is the opposite behavior of the yield of a bond, where market demand will increase the price of the bond and therefore decrease its yield.

Figure 2: Euribor Spreads versus 6M



To construct the projection curve, the OIS instruments are used to determine the base curve and then the LIBOR-based instruments are applied. In some economies the situation is more complex than described here, but the complications are only technical in nature and there is no barrier to constructing the relevant curves.

Since discounting and projection can't be determined using a single curve, the question naturally arises as to which curve should be used to discount. Two different answers emerged:

- Theoretical – The LIBOR was previously considered to be the risk-free rate. The body of quantitative finance tells us we must discount at the risk-free rate when pricing derivative securities. Whereas LIBOR involved some credit-worthiness of the banks that quote LIBOR, the overnight rate is the actual rate traded in the market for uncollateralized lending between banks for one day. Because of this, the overnight rate is now recognized as the best proxy for the risk-free rate.
- Funding – The time value of money is closely connected to how a firm obtains the money to meet future obligations.

When a swap is perfectly collateralized in cash, the two schools of thought agree that the OIS is the correct curve to use for discounting. Therefore, we will use Equation 3 for the remainder of the paper.

Valuation

The terminology of a swap is determined by which party pays the fixed leg. A swap is called a "payer" swap if you are the party paying the fixed leg. A swap is called a "receiver" swap if you are the party paying the floating leg and therefore receiving the fixed leg.

The value of an interest rate swap is the difference between the paying leg and the receiving leg. Fundamentally, the legs are no different from other financial instruments; each coupon payment is the present value of the product of a principal, an accrual fraction, and a coupon rate. For interest rate swaps, the principal is referred to as a notional principal amount (NPA) due to the fact that this amount is never exchanged.[6] If the principal were paid, it would have no impact on the valuation of an interest rate swap, as both legs would pay this amount at maturity and thereby exactly offset the value from the other leg.

The current market value of a receiver swap is given in Equation 4:

Equation 4:

$$\mathit{Receiver\ Swap\ Current\ Market\ Value} = V(t,\tau,c) = \sum_{i=1}^{N} D(t,T_i)\cdot N\cdot\ \tau_i\cdot c - \sum_{j=1}^{M} D(t,T_j)\cdot N\cdot\ \tau_j\cdot\left(L_j^{\tau}(t)+\kappa^{\tau}\right)$$

Where:

- $D(t,T_i)$ = The discount factor, representing the price of a zero coupon bond at the time it is bought (t) and the time it matures (T_i)
- N = Number of payments on the fixed leg
- T_i = Paying leg payment frequency
- c = Coupon
- $D(t,T_j)$ = The discount factors determined by the current interest rate environment
- τ_j = Receiving leg payment frequency
- $L_j^{\tau}(t)$ = The expected index rates determined by the current interest rate environment
- κ^{τ} = Spread above LIBOR implied by the overnight curve

The set of dates will generally be different for the fixed leg and the floating leg. This means the paying leg and receiving leg have different payment frequencies. Typically, the floating leg is more frequent, every other floating leg payment date lines up with the corresponding fixed leg payment date, and the maximum values of τ_i and τ_j are equal $(T_N = T_M)$.

For instance, USD swaps are semi-annual fixed leg payments swapped for quarterly floating leg payments. The actual frequency depends on the currency of the swap. The major conventions for the major currencies are shown in Table 2:

Table 2: Swap Conventions for Major Jurisdictions[7]

Jurisdiction	Fixed Leg Frequency	Fixed Leg Day Count	Floating Leg Frequency	Floating Leg Day Count
USD	6M	30/360	3M	ACT/360
EUR (>1Y)	1Y	30/360	6M	ACT/360
GBP (>1Y)	6M	ACT/365	6M	ACT/365
JPY	6M	ACT/365	6M	ACT/360
CHF (>1Y)	1MY	30/360	6M	ACT/360

At trade execution, both legs will price close to par. The swaps will be issued at the “par swap rate”[8], which is the unique value of the coupon in Equation 4 that leads to a swap having zero value (V = 0). The result can be thought of as a type of average value of the prevailing interest rates, as shown in Equation 5:

6 Since this principal is never exchanged, it is only “notionally” the principal, meaning “in name only.” Although in practice, “notional” and “principal” are usually used interchangeably.
7 OpenGamma, 'Interest Rate Instruments and Market Conventions Guide', 2013, http://www.opengamma.com/sites/default/files/interest-rate-instruments-and-market-conventions.pdf.
8 The par swap rate can be found by setting Equation 4 to zero (with no spread (κ) present) and solving for the value of c.

Equation 5:

$$An\ Average\ of\ the\ Prevailing\ Interest\ Rate = s(t,T) = \frac{\Sigma_{j=1}^{M} D(t,T_j)\cdot\ \tau_j \cdot\ L_j^{\tau}(t)}{\Sigma_{i=1}^{M} D(t,T_i)\cdot\tau_i}$$

Where:

- (t, T_j) = The discount factors determined by the current interest rate environment
- τ_j = Receiving leg payment frequency
- $L_j^{\tau}(t)$ = The expected index rates determined by the current interest rate environment
- $D(t, T_i)$ = The discount factor, representing the price of a zero coupon bond at the time it is bought (t) and the time it matures (T_i)
- τ_i = Paying leg payment frequency

Steps to value the floating leg of a swap:

1. Determine the rates from the projection curve (with careful treatment of date conventions)
2. Add spread
3. Calculate the coupon time fractions
4. Calculate Discount factors (usually involves an interpolation[9])
5. Multiply all together

Steps to value the fixed leg of a swap:

1. Calculate coupon time periods
2. Calculate discount factors (usually involves an interpolation[10])
3. Multiply by the fixed coupon rate

The value of the swap is the value of the leg you pay subtracted from the value of the leg you receive. This results in the "dirty price" of the swap. To calculate the "clean price" of the swap, subtract the accrued interest from the dirty price of each leg, and the clean price of the swap is the difference of the clean prices of the legs.

Accrued Interest

The accrued interest represents the fraction of the upcoming cash flow that is owed to the current swap counterparty if one of those counterparties was to change during the lifetime of the contract. Accrued interest is calculated the same way for swaps as it is for bonds, as both fixed and floating cash flows are already known during the cash flow period.

Together with the magnitude of the fixed rate cash flows, this quantity can be calculated by both counterparties as the only part of the swap calculation that both counterparties can agree on to arbitrary precision. The accrued interest is calculated as shown in Equation 6:

9 See Appendix B
10 See Appendix B.

Equation 6:

$$Accrued\ Interest = \frac{\alpha(t, T_{n-1})}{\tau_n} CF_n$$

Where:

- $\alpha(t, T_{n-1})$ = The day count fraction between the accrual end date of the last coupon period, where t is the day count of the current cash flow period and T_{n-1} is the last coupon period's accrual end date
- τ_n = The day count of the current cash flow period
- CF_n = The cash flow associated with the current coupon period

Since these two quantities are always calculated in the same day count convention, the denominators for most accrual conventions will cancel and the accrued interest simplifies to Equation 7:

Equation 7:

$$Accrued\ Interest\ (Simplified) = \frac{d}{D} CF_n$$

Where:

- d = The number of days since the start of the current accrual period
- D = The number of days in the current accrual period
- CF_n = The cash flow associated with the current coupon period

Therefore, the fraction becomes the ratio of the number of days that have passed since the last accrual end date and the number of days in the current cash flow period (as calculated in the numerator of the day count function).

Valuation Example

Suppose a bank sells (pays the fixed rate) a collateralized three-year 0.95% swap with a principal of €1,000,000 on 4-Jun-2014. Since the swap is denominated in euros, the fixed leg is annual 30/360 and the floating leg is semi-annual ACT/360, as shown in Table 2. In the following sections, we calculate the value of the swap on the settlement date (the date on which the valuation takes place) of 23-Jun-2014.

Fixed Leg

The fixed leg is the easier of the two legs to value since the coupon does not need to be estimated. The pricing is therefore similar to pricing a fixed rate bond; simply discount the known cash flows. The difference is that a bond is calculated using a flat yield, whereas a swap is always valued "off a curve," meaning the discounting is done with the prevailing zero coupon bond. Another crucial difference is that most bonds have level coupons, where the coupons are the same for all cash flow periods, regardless of whether they contain the same number of days. Fixed legs of swaps do not have level coupons; the magnitude of the coupon depends on the accrual convention and business day adjustment rules stated in the contract:

Table 3: Fixed Leg Payment Schedule

Date	Coupon Rate	Total Coupon Payment	Discount Factor	PV Of Payment
4-Jun-2015	0.95%	€9,500.00	0.999582613	€9,496.00
4-Jun-2016	0.95%	€9,500.00	0.998771416	€9,488.00
4-Jun-2017	0.95%	€1,009,500.00	0.996254447	€1,005,719.00

The sum of the present value (PV) of the payments gives the dirty price of the fixed leg.

The accrued interest for the fixed leg is calculated using a 30/360 day count method. For this swap, the accrual fraction is particularly simple – one year is always 1.0 in the 30/360 day count method. Therefore, we only need to count the number of days since the last coupon payment up to, but not including, the settlement date. In this example, there are 19 days since the last coupon payment (4-Jun- 2014) and the settlement date (23-Jun-2014). Therefore, the accrued interest is equal to 19/360 multiplied by the total coupon payment, resulting in €501.00 of accrued interest as of the settlement date.

Floating Leg

Valuing the floating leg is similar to valuing the fixed leg. However, to value a floating leg the coupon must be determined from the prevailing yield curve:

Date	Coupon Rate	Total Coupon Payment	Discount Factor	PV Of Payment
4-Dec-2014	0.386%	€1,962.17	0.999773010	€1,962.00
4-Jun-2015	0.445%	€2,249.72	0.999582762	€2,249.00
4-Dec-2015	0.445%	€2,262.08	0.999270810	€2,260.00
44-Jun-2016	0.232%	€1,179.33	0.998771721	€1,178.00
4-Dec-2016	0.232%	€1,179.33	0.997834606	€1,177.00
4-Jun-2017	0.641%	€1,003,240.61	0.996254908	€999,483.00

The sum of the PV of the payments gives the dirty value of the floating leg.

The accrued interest is also calculated in a similar fashion as the fixed leg; however, the floating leg day count method is ACT/360. The numerator is the number of days since the beginning of the current coupon period and the settlement date. In this example, there are 19 days in the period, the same as the fixed leg in this case. The denominator is the actual number of days in the current coupon period. The current coupon period begins on 4-Jun-2014 and ends on 4-Dec-2014, which gives 182 days in the period. Therefore, the accrued interest is 19/182 of the total coupon, resulting in €204.00 of accrued interested as of the settlement date.

Net Value of a Swap

The net value is calculated by taking the difference of the present value of the two legs of the swap. Since the bank sold this swap, they are the fixed leg payer. From the bank's perspective, the value is the difference of the floating leg (asset) and the fixed leg (liability). This net value is the dirty price of the swap, since it is based on total cash flows. From the corporate's perspective, who bought the swap, they are the fixed leg receiver. The net value from the corporate's point of view is the difference of the fixed leg (asset) and the floating leg (liability). There is a portion of the current cash flow that would need to be paid to the seller of the swap, if it were sold. As stated above, the clean price is the difference between the net value and the accrued interest. The clean price of the swap is calculated by first calculating the clean price of each leg and subsequently performing the same difference.

Note that the present value of the notional exactly cancels when the final difference is taken. Due to this fact, the notional is not exchanged at inception or maturity of an interest rate swap (hence the terminology "notional principal amount").

Value of a Swap Over Time

Immediately after the trade execution the swap will no longer have zero value, even if the interest rate environment remains unchanged due to interest accruing and coupons being paid. For example, half of the coupon will have accrued one quarter into a swap contract that pays semi-annually. Recall that this amount of money is due to the holder of the swap upon transfer of this contract to a third party. Therefore, even though the future cash flows net to zero for a par swap, it no longer has a zero market value.

If the interest rate environment changes (i.e., if there are changes in the market quotes that determine the yield curve, such as the cash rates, futures, and swap rates traded in the market) then the floating rate coupons used to determine the par swap will not be the value of the floating rate coupons paid throughout the lifetime of the swap. Swaps are said to be "in-the-money" for the owner of the side of the swap where the interest rate movements are favorable and the contract becomes an asset. For example, when interest rates decrease the fixed rate receiver (the floating rate payer) is in-the-money since their forecasted payments are greater than their actual payments. Conversely, for the fixed rate payer (the floating rate receiver) the swap is "out-of-the-money." This can be determined by calculating the par swap rate for the remaining tenor of the swap compared to the fixed rate in the contract:

Table 5: Moneyness Definitions for Swaps

	Current Par Swap > Fixed Rate	Current Par Swap < Fixed Rate
Fixed Rate Payer	In the money	Out of the money
Fixed Rate Receiver	Out of the money	In the money

Intuitively, if the current par swap rate is higher than the par swap rate at swap initiation, the party paying the contracted fixed rate is better off, whereas if the current par swap rate is lower, the party receiving the contracted fixed rate is better off.

Summary

This paper summarizes the important aspects of swap valuation, from the calculation of the cash flows to the determination of market value from swap initiation to maturity. While there are many similarities to pricing bonds, the subtle differences can lead to mispricing, incorrect attribution, and counterparty disputes. Furthermore, this paper highlights the importance of incorporating non-zero LIBOR spreads that appeared in the LIBOR quotes after the financial crisis of 2008.

Appendix A

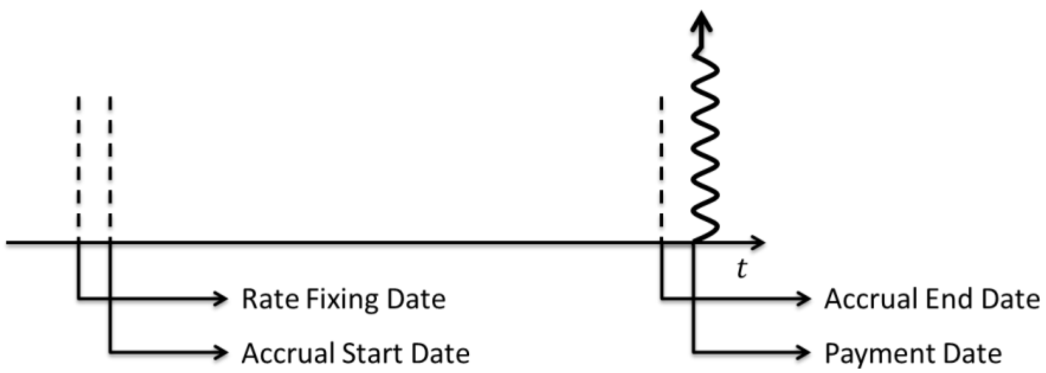
Getting the Dates Right

When performing the valuation of an interest rate swap versus a counterparty, it is crucial to agree on the rate dates. Rates have four important dates:

- The date on which the expectation is taken
- The date on which the rate is fixed for the cash flow
- The accrual start date
- The accrual end date

For coupons, the date on which the cash flow is paid is also included. For interest rate swaps, this is usually very close to the accrual end date.

Figure 3: Schematic Timeline of Important Dates



Appendix B

Interpolation with Natural Cubic Spline

The assumptions in the natural cubic spline are that the functional form of the interpolating function is cubic with coefficients determined by three constraints: continuity, smoothness, and continuity of the second derivative. This produces an underdetermined problem, which is solved by the further constraint that the second derivative at the first and last known point is zero. Other splines exist, such as the “clamped spline,” where the first derivatives are specified at the boundary as well as the “financial spline,” where the first derivative is set to zero at the boundary (this naturally lends itself to a flat rate extrapolation).

Appendix C

Definitions

Table 6: Common Terms and Definitions

Term	Definition
Business Center	The holiday calendar that is relevant for aspects of the contract. For instance, the payment business center indicates which days are allowable for determining when payments occur, whereas the rate fixing business center will determine which days are allowable for determining when the rate fixing can occur. The latter usually includes London because the LIBOR rates are fixed there and they cannot be fixed on a London bank holiday.
Business Day Adjustment	Indicates how the relevant date (i.e., payment date, rate fixing date) is adjusted if the calculated date falls on a bank holiday in the relevant business center. The standard is "modified following," which means that the relevant date will be the next good business day after the calculated date, unless that adjustment lands in the next calendar month, in which case the preceding good business day is chosen. Other possibilities are following, previous, and modified previous. For instance, if the swap pays on the 29th of each month, and 29-Apr-2011 was determined to be a London bank holiday due to the royal wedding, LIBOR cannot fix on that date. The next good business day in London was Monday, 3-May-2011. Note this fell into the next month, so any contract specifying "modified following" would use 28-Apr-2011 as the next roll date.
Calculation Agent	Specifically who determines the floating rate as well as where and when the observation for the floating rate index will take place. This section also specifies how to determine the rate during a pricing disruption event. The term "determining party" is similar; however, there could be a slightly different legal interpretation between the two.
Day Count Basis	The method of calculating the length of time between two dates. Typical values are ACT/360, where you take the "actual" number of days between two dates and divide by 360. Note this will give a day count fraction greater than 1.0 for one year. Other methods include ACT/365 and 30/360. The day count basis is also known as day count fraction.[11]
Fixed Rate	The coupon rate that is not pegged to a financial observable index.
Leg	One of the streams of cash flows. A typical vanilla interest rate swap has a fixed leg and a floating leg. Other legs exist, such as "structured" legs, where the cash flow is determined by a complex formula of financial observables, such as interest rate indices and foreign exchange rates.
Rate Fixing Dates	The date on which a rate for the upcoming coupon period is set. The contract will specify how this is set by specifying a calculation agent and a series of prescriptions/mitigations to follow if the calculation agent is unavailable. Also known as roll date.
Roll Date	Another name for rate fixing date. The term roll date is favored in the industry.

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OpenGamma. *"Interest Rate Instruments and Market Conventions Guide"*, 2013. http://developers.opengamma.com/quantitative-research/Interest-Rate-Instruments-and-MarketConventions.pdf.

11 See 2006 ISDA Definitions. (New York, NY: International Swaps and Derivatives Association, Inc., 2006) for comprehensive descriptions of day count conventions.